"Deformation Gradient" (Con'd)

Example

Three line elements: $\mathrm{d}\boldsymbol{x}_x = \boldsymbol{X}_x\,\mathrm{d}x,\ \mathrm{d}\boldsymbol{x}_y = \boldsymbol{X}_y\,\mathrm{d}y,\ \mathrm{d}\boldsymbol{x}_z = \boldsymbol{X}_z\,\mathrm{d}z$

Definition: $\mathrm{d}V_0 = (\mathrm{d}\boldsymbol{x}_x \times \mathrm{d}\boldsymbol{x}_y)\cdot\mathrm{d}\boldsymbol{x}_z = (\mathrm{d}\boldsymbol{X}_x \times \mathrm{d}\boldsymbol{X}_y)\cdot\mathrm{d}\boldsymbol{X}_z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

Calculation: $\mathrm{d}V = (\mathrm{d}\boldsymbol{X}_x \times \mathrm{d}\boldsymbol{X}_y)\cdot\mathrm{d}\boldsymbol{X}_z = [(\mathbf{F}\cdot\mathrm{d}\boldsymbol{x}_x)\times(\mathbf{F}\cdot\mathrm{d}\boldsymbol{x}_y)]\cdot\mathbf{F}\cdot\mathrm{d}\boldsymbol{x}_z$

$\mathrm{d}V = \det\mathbf{F}\,(\mathrm{d}\boldsymbol{x}_x \times \mathrm{d}\boldsymbol{x}_y)\cdot\mathrm{d}\boldsymbol{x}_z = \det\mathbf{F}\mathrm{d}V_0$

Conclusions

$$\boxed{\det\boldsymbol{F} = \frac{\mathrm{d}V}{\mathrm{d}V_0} > 0}$$

and

$$\boxed{\det\boldsymbol{F} = 1 \text{ if } \mathrm{d}V = \mathrm{d}V_0}$$

$\Rightarrow$ isochoric behavior (incompressibility condition)

Relation between $\boldsymbol{F}$ and $\boldsymbol{u}$

with

$$\boldsymbol{X} = \boldsymbol{x} + \boldsymbol{u}, \quad \nabla_{\boldsymbol{x}} \boldsymbol{x} = \boldsymbol{I} \;\Rightarrow\; \nabla \boldsymbol{X} = \boldsymbol{I} + \nabla \boldsymbol{u}$$

one gets

$$\Rightarrow \; (\nabla \boldsymbol{X})^{\mathsf{T}} = \boldsymbol{I}^{\mathsf{T}} + (\nabla \boldsymbol{u})^{\mathsf{T}} = \boldsymbol{I} + (\nabla \boldsymbol{u})^{\mathsf{T}} = \boldsymbol{F}$$

Normal Strains

Line element in the reference configuration

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}L\boldsymbol{m}, \quad (|\boldsymbol{m}| = 1)$$

Line element in the actual configuration

$$\mathrm{d}\boldsymbol{X} = \mathrm{d}l\widetilde{\boldsymbol{m}}, \quad (|\widetilde{\boldsymbol{m}}| = 1, \ \widetilde{\boldsymbol{m}} \neq \boldsymbol{m})$$

Definition

$$\varepsilon_{mm} = \frac{\mathrm{d}l - \mathrm{d}L}{\mathrm{d}L} = \frac{\mathrm{d}l}{\mathrm{d}L} - 1$$

ε_{mm} - normal strain in the neighborhood of P in direction of $\boldsymbol{m}$

Green-Lagrange Strain Tensor

Square length of the line elements

$$\mathrm{d}l^2 = \mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{X} = \mathrm{d}\boldsymbol{X} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X} = \mathrm{d}L^2\, \boldsymbol{m} \cdot \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} \cdot \boldsymbol{m}$$

$$\left(\frac{\mathrm{d}l}{\mathrm{d}L}\right)^2 = \boldsymbol{m} \cdot \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} \cdot \boldsymbol{m} = (\varepsilon_{mm} + 1)^2 = \varepsilon_{mm}^2 + 2\varepsilon_{mm} + 1 \approx 2\varepsilon_{mm} + 1$$

$$\begin{aligned} \varepsilon_{mm} &= \frac{1}{2}\boldsymbol{m} \cdot \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} \cdot \boldsymbol{m} - \frac{1}{2} = \frac{1}{2}\boldsymbol{m} \cdot \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} \cdot \boldsymbol{m} - \frac{1}{2}\boldsymbol{m} \cdot \boldsymbol{I} \cdot \boldsymbol{m} \\ &= \boldsymbol{m} \cdot \boxed{\frac{1}{2}\left(\boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} - \boldsymbol{I}\right)} \cdot \boldsymbol{m} = \boldsymbol{m} \cdot \boxed{\boldsymbol{G}} \cdot \boldsymbol{m} \end{aligned}$$

$\boldsymbol{G}$ - Green-Lagrange strain tensor

Shear

Two line elements in the reference configuration

$$\mathrm{d}\boldsymbol{x}_n = \mathrm{d}L_n\,\boldsymbol{n},\ \mathrm{d}\boldsymbol{x}_p = \mathrm{d}L_p\,\boldsymbol{p}$$

Two line elements in the actual configuration

$$\mathrm{d}\boldsymbol{X}_{\tilde{n}} = \mathrm{d}l_{\tilde{n}}\,\widetilde{\boldsymbol{n}},\ \mathrm{d}\boldsymbol{X}_{\tilde{p}} = \mathrm{d}l_{\tilde{p}}\,\widetilde{\boldsymbol{p}}$$

Shear strain

$$\gamma_{np} = \frac{\pi}{2} - \alpha_{np}$$

Some calculations

$$\mathrm{d}\boldsymbol{X}_{\tilde{n}} \cdot \mathrm{d}\boldsymbol{X}_{\tilde{p}} = \mathrm{d}l_{\tilde{n}}\,\mathrm{d}l_{\tilde{p}}\,\cos\alpha_{np} = \mathrm{d}l_{\tilde{n}}\,\mathrm{d}l_{\tilde{p}}\,\sin\gamma_{np}$$
$$\mathrm{d}\boldsymbol{X}_{\tilde{n}} \cdot \mathrm{d}\boldsymbol{X}_{\tilde{p}} = (1+\varepsilon_{nn})\,(1+\varepsilon_{pp})\sin\gamma_{np}\,\mathrm{d}L_n\,\mathrm{d}L_p$$

Shear (Con'd)

Assumption

$\gamma_{np} \ll 1 \Rightarrow \sin\gamma_{np} \approx \gamma_{np}$

$$(1+\varepsilon_{nn})(1+\varepsilon_{pp})\sin\gamma_{np} \approx (1+\varepsilon_{nn}+\varepsilon_{pp}+\varepsilon_{nn}\varepsilon_{pp})\gamma_{np} \approx \gamma_{np}$$

$$\mathrm{d}\boldsymbol{X}_{\tilde{n}} \cdot \mathrm{d}\boldsymbol{X}_{\tilde{p}} = \mathrm{d}\boldsymbol{x}_n \cdot \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x}_p = \mathrm{d}L_n \mathrm{d}L_p \boldsymbol{n} \cdot \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} \cdot \boldsymbol{p} = \gamma_{np} \mathrm{d}L_n \mathrm{d}L_p$$

$\Rightarrow \quad \gamma_{np} = \boldsymbol{n} \cdot \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} \cdot \boldsymbol{p}$
with $\boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} = 2\boldsymbol{G} + \boldsymbol{I}$

$\Rightarrow \quad \gamma_{np} = 2\boldsymbol{n} \cdot \boldsymbol{G} \cdot \boldsymbol{p}$

Cauchy Strain Tensor

Linearization

$$\boldsymbol{G} = \frac{1}{2}\left(\boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} - \boldsymbol{I}\right)$$

$$\boldsymbol{F} = \boldsymbol{I} + (\boldsymbol{\nabla u})^{\mathsf{T}}, \quad \boldsymbol{F}^{\mathsf{T}} = \boldsymbol{I} + \boldsymbol{\nabla u}$$

$$\boldsymbol{G} = \frac{1}{2}\left\{[\boldsymbol{I} + \boldsymbol{\nabla u}] \cdot \left[\boldsymbol{I} + (\boldsymbol{\nabla u})^{\mathsf{T}}\right] - \boldsymbol{I}\right\}$$

$$\boldsymbol{G} = \frac{1}{2}\left[\boldsymbol{I} + \boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^{\mathsf{T}} + \boldsymbol{\nabla u} \cdot (\boldsymbol{\nabla u})^{\mathsf{T}} - \boldsymbol{I}\right]$$

$$\boldsymbol{G} = \frac{1}{2}\left[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^{\mathsf{T}} + \boldsymbol{\nabla u} \cdot (\boldsymbol{\nabla u})^{\mathsf{T}}\right]$$

$$\boldsymbol{G} \approx \boxed{\frac{1}{2}\left[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^{\mathsf{T}}\right] = \boldsymbol{\varepsilon}}$$

Components of the Cauchy Strain Tensor

normal strains

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \ \varepsilon_{yy} = \frac{\partial u}{\partial y}, \ \varepsilon_{zz} = \frac{\partial u}{\partial z}$$

shear strains

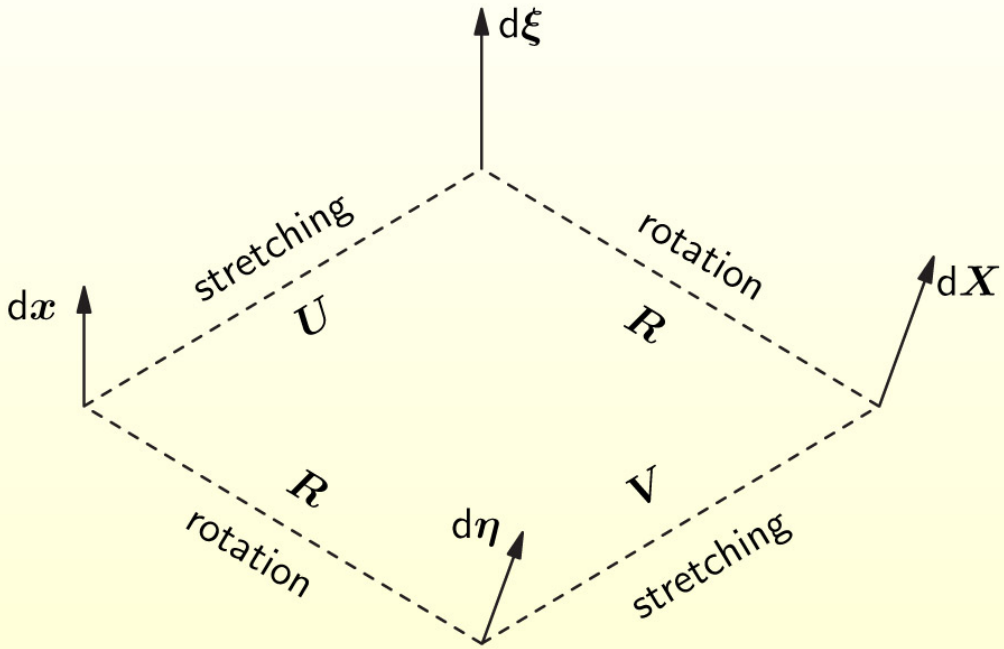
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}, \ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}, \ \gamma_{yz} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}$$

corresponding tensor components

$$\frac{1}{2}\gamma_{xy} = \varepsilon_{xy}, \ \frac{1}{2}\gamma_{xz} = \varepsilon_{xz}, \ \frac{1}{2}\gamma_{yz} = \varepsilon_{yz}$$

Note: $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathsf{T}}$

Polar Decomposition



$$
\begin{aligned}
\boldsymbol{F} &= \boldsymbol{R} \cdot \boldsymbol{U} \\
&= \boldsymbol{V} \cdot \boldsymbol{R} \\
\boldsymbol{C} &= \boldsymbol{U}^2 = \boldsymbol{F}^\mathsf{T} \cdot \boldsymbol{F} \\
\boldsymbol{B} &= \boldsymbol{V}^2 = \boldsymbol{F} \cdot \boldsymbol{F}^\mathsf{T}
\end{aligned}
$$

Stresses

Basics of Kinetics

Classification of the External Loading

Types of loadings

1. Natural Models
   - body / mass / volume loading (forces, moments)
   - surface / contact loading (forces, moments)
2. In Addition, Two Artificial Loading Models
   - line loading (forces, moments)
   - single point loading (forces, moments)

Dimensional Analysis

- $[\mathbf{F}] = \mathsf{N}, \quad [\mathbf{M}] = \mathsf{Nm}$
- body loading: per volume
- surface loading: per area
- line loading: per line

Body Loading

Body force $\rho\,(\boldsymbol{X},t)\,\boldsymbol{k}\,(VX,t) = \boldsymbol{k}^{\mathsf{V}}\,(\boldsymbol{X},t)$

By analogues $\rho\,(\boldsymbol{X},t)\,\boldsymbol{l}\,(VX,t) = \boldsymbol{l}^{\mathsf{V}}\,(\boldsymbol{X},t)$ body moment

Examples

- weight force:
  $\rho\,\boldsymbol{k} = -\rho g \boldsymbol{e}_3$
- inertia force:
  $\rho\,\boldsymbol{k} = -\rho\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{X})$
- potential force:
  $\rho\,\boldsymbol{k} = -\rho\boldsymbol{\nabla}\Pi\,(\boldsymbol{X}t)$

Stresses

Stress vector:

$$\boldsymbol{t} = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{f}}{\Delta A}$$

Couple stress vector:

$$\boldsymbol{M} = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{m}}{\Delta A}$$

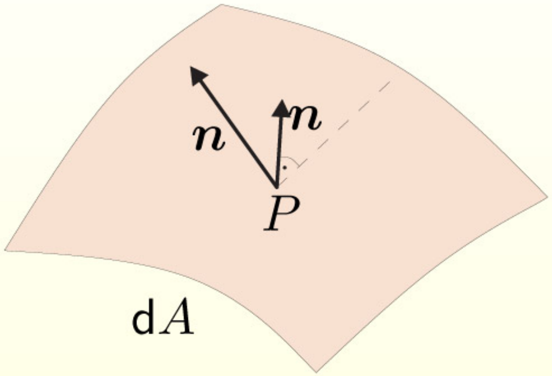
Resultant force:

$$\boldsymbol{f}^{\mathsf{R}} = \int_V \rho \boldsymbol{k} \, \mathrm{d}V + \int_A \boldsymbol{t} \, \mathrm{d}A$$

Resultant moment:

$$\boldsymbol{m}_0^{\mathsf{R}} = \int_V \rho \left(\boldsymbol{l} + \boldsymbol{r} \times \boldsymbol{k} \right) \mathrm{d}V + \int_A \left(\boldsymbol{M} + \boldsymbol{r} \times \boldsymbol{t} \right) \mathrm{d}A$$

Stress Tensor



Two vectors

$\boldsymbol{t}$ – stress vector

$\boldsymbol{n}$ – normal to the surface

Components of the stress vector

$$\boldsymbol{t} = t_{\boldsymbol{n}}\boldsymbol{n} + t_{\boldsymbol{t}}\boldsymbol{e}_t = t_{\boldsymbol{n}}\boldsymbol{n} + t_{t_1}\boldsymbol{e}_{t_1} + t_{t_2}\boldsymbol{e}_{t_2}$$

where $t_{t_1}\boldsymbol{e}_{t_1}$ and $t_{t_2}\boldsymbol{e}_{t_2}$ are arbitrary tangential directions in the surface

$$\boldsymbol{n} \perp \boldsymbol{e}_{t_1}, \ \boldsymbol{n} \perp \boldsymbol{e}_{t_2}, \ \boldsymbol{e}_{t_1} \perp \boldsymbol{e}_{t_2}$$

so $\boldsymbol{n}$, $\boldsymbol{e}_{t_1}$ and $\boldsymbol{e}_{t_2}$ form an arbitrary orthonormal base

Cauchy's Lemma

$$\boldsymbol{t}(\boldsymbol{r}, \boldsymbol{n}, t) = \boldsymbol{n} \cdot \boldsymbol{T}(\boldsymbol{r}, t)$$

Equilibrium (Static Case)

Only forces!

$$\int_V \rho \boldsymbol{k} \, \mathrm{d}V + \int_A \boldsymbol{t} \, \mathrm{d}A = \boldsymbol{0}$$

$$\int_V (\boldsymbol{r} \times \rho \boldsymbol{k}) \, \mathrm{d}V + \int_A (\boldsymbol{r} \times \boldsymbol{t}) \, \mathrm{d}A = \boldsymbol{0}$$

Divergence theorem (Gauß-Ostrogradsky)

$$\int_A \boldsymbol{t} \, \mathrm{d}A = \int_A \boldsymbol{n} \cdot \boldsymbol{T} \, \mathrm{d}A = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{T} \, \mathrm{d}A \quad \Longrightarrow \quad \int_V (\rho \boldsymbol{k} + \boldsymbol{\nabla} \cdot \boldsymbol{T}) \, \mathrm{d}V = \boldsymbol{0}$$

Local form

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} + \rho \boldsymbol{k} = \boldsymbol{0} \quad \Longleftrightarrow \quad \mathrm{div} \boldsymbol{T} + \rho \boldsymbol{k} = \boldsymbol{0} \quad \Longleftrightarrow \quad T_{ij,i} + \rho k_j = 0_j$$

D'Alambert's Principle

Only forces, but inertia is considered!

$$\int\limits_V \rho \boldsymbol{k} \,\mathrm{d}V + \int\limits_A \boldsymbol{t} \,\mathrm{d}A - \int\limits_V \rho \ddot{\boldsymbol{X}} \,\mathrm{d}V = \boldsymbol{0}$$

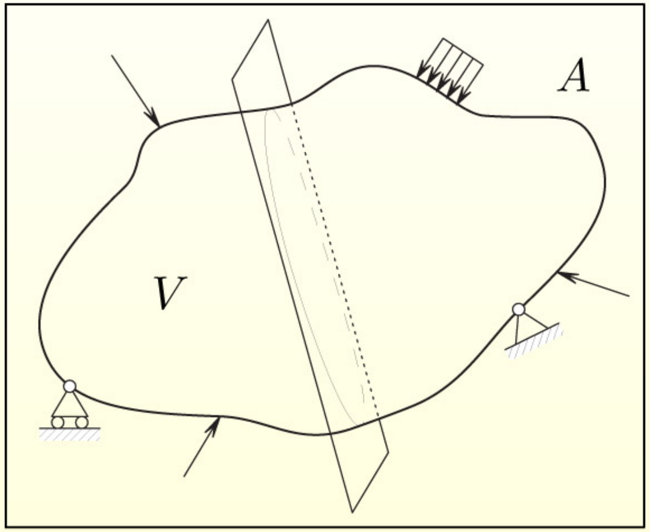
Divergence theorem (Gauß-Ostrogradsky)

$$\int\limits_A \boldsymbol{t} \,\mathrm{d}A = \int\limits_A \boldsymbol{n} \cdot \boldsymbol{T} \,\mathrm{d}A = \int\limits_V \boldsymbol{\nabla} \cdot \boldsymbol{T} \,\mathrm{d}A \quad \Longrightarrow \quad \int\limits_V \left(\rho \boldsymbol{k} + \boldsymbol{\nabla} \cdot \boldsymbol{T} - \rho \ddot{\boldsymbol{X}} \right) \mathrm{d}V = \boldsymbol{0}$$

Local form

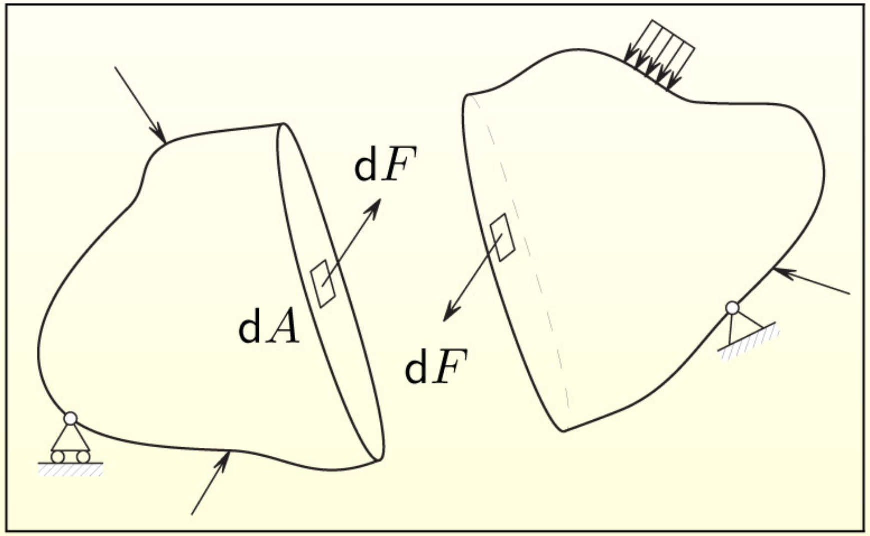
$$\boldsymbol{\nabla} \cdot \boldsymbol{T} + \rho \boldsymbol{k} = \rho \ddot{\boldsymbol{X}} \quad \Longleftrightarrow \quad T_{ij,i} + \rho k_j = \rho \ddot{X}_j$$

Continuum Mechanics - Basics



- cutting principle (method of sections)
- axiom of reciprocal action (Newton's Third Law)
- Continuum Mechanics governing equations

Continuum Mechanics - Non-polar



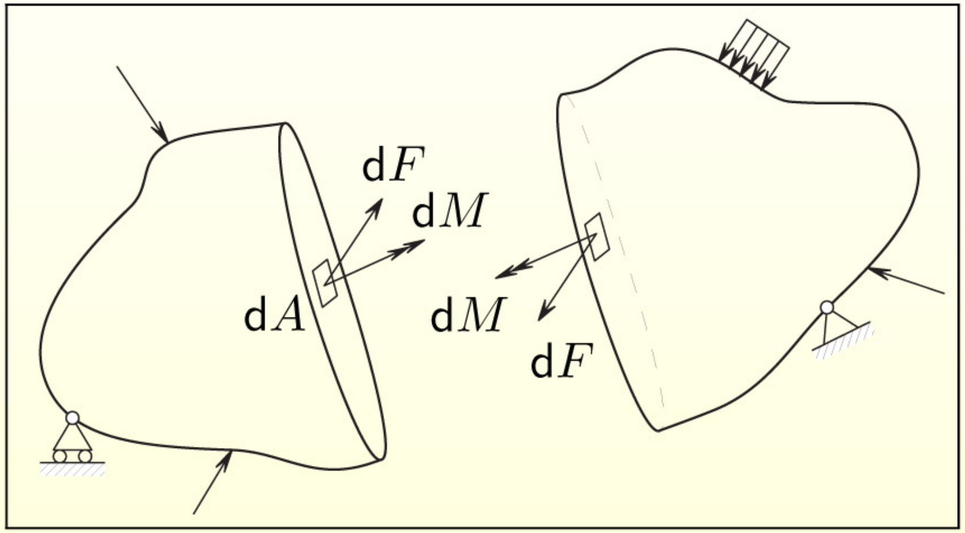
- only force actions
- symmetric stress tensor
- only translations

Basic Course Engineering Mechanics

- Static equilibrium
  - Forces
  - Moments
- Dynamic equilibrium
  - Balance of momentum
  - Balance of moment of momentum
- Dependent or independent relations?[25]

[25] Truesdell, C. (1964). Die Entwicklung des Drallsatzes. *ZAMM* **44**(4/5):149–158

Continuum Mechanics - Polar



- force and moments actions
- symmetric and nonsymmetric stress tensors
- translations and rotations (independent!)

Balances

General Statements

Basic Assumptions

Thermodynamics

- Equilibrium Thermodynamics
- Non-equilibrium Thermodynamics

4 Laws of Thermodynamics

- 1st Law – Energy Balance
- 2nd Law – Entropy Balance (Process Direction)
- 3rd Law – $\Theta = 0K \iff S = 0$
- 4th Law – Equilibrium of Systems

State Variables

- macroscopic
- measurable
- independent

Phenomenological Variables

Extensive (Additive) Variables

- e.g., proportional to the mass
- example: inner energy, which depends only on the kinematics and the temperature

Intensive Variables

- e.g., not proportional to the mass
- examples: density, temperature

General Balance Equation

$\Psi(\boldsymbol{X}, t)$ and $\Psi_0(\boldsymbol{x}, t)$ specific scalar properties distributed in dV or dV_0
Integration over all body points results in $Y(t)$

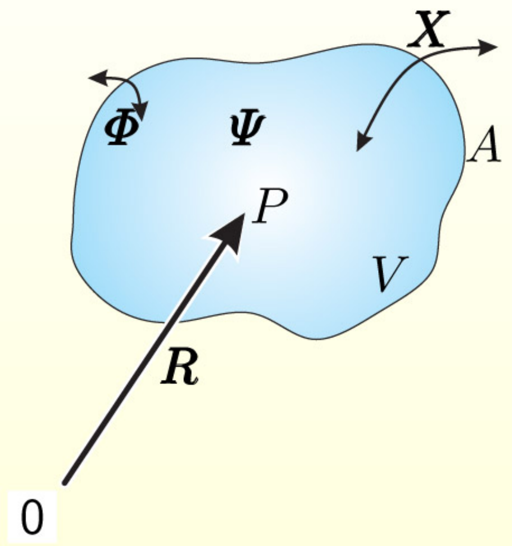
$$Y(t) = \int\limits_V \Psi(\boldsymbol{X}, t) dV = \int\limits_{V_0} \Psi_0(\boldsymbol{x}, t) dV_0$$

With $dV = (\det \boldsymbol{F}) dV_0$ one gets $\Psi_0(\boldsymbol{x}, t) = (\det \boldsymbol{F}) \Psi(\boldsymbol{X}, t)$

$$\frac{D}{Dt} Y(t) = \frac{D}{Dt} \int\limits_V \Psi(\boldsymbol{X}, t) dV = \int\limits_A \Phi(\boldsymbol{X}, t) dA + \int\limits_V \Xi(\boldsymbol{X}, t) dV$$

$$\frac{D}{Dt} Y(t) = \frac{D}{Dt} \int\limits_{V_0} \Psi_0(\boldsymbol{x}, t) dV_0 = \int\limits_{A_0} \Phi_0(\boldsymbol{x}, t) dA_0 + \int\limits_{V_0} \Xi_0(\boldsymbol{x}, t) dV_0$$

General Formulation



$\boldsymbol{\Psi}$ - balance variable
$\boldsymbol{\Phi}$ - action through the surface (flux)
$\boldsymbol{X}$ - action onto the volume (surface)

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \boldsymbol{\Psi}(\boldsymbol{R},t)\,\mathrm{d}V = \int_A \boldsymbol{\Phi}(\boldsymbol{R},t)\,\mathrm{d}A + \int_V \boldsymbol{X}(\boldsymbol{R},t)\,dV$$
$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{V_0} \boldsymbol{\Psi_0}(\boldsymbol{r},t)\,\mathrm{d}V_0 = \int_{A_0} \boldsymbol{\Phi_0}(\boldsymbol{r},t)\,\mathrm{d}A_0 + \int_{V_0} \boldsymbol{X_0}(\boldsymbol{r},t)\,\mathrm{d}V_0$$

Comments Concerning the General Formulation

- $\boldsymbol{\Phi}$ - action through the surface A, property of the surface A: orientation $\boldsymbol{n}$
  $\boldsymbol{\Phi}(\boldsymbol{R},t)\boldsymbol{\Phi}(\boldsymbol{R},t) \Rightarrow \boldsymbol{\Phi}(\boldsymbol{R},\boldsymbol{n},t)$
- Cauchy´s theorem is valid
  ${}^{(n)}\boldsymbol{\Phi}(\boldsymbol{R},t)\boldsymbol{\Phi}(\boldsymbol{R},t) = \boldsymbol{n}^{(n+1)} \cdot \boldsymbol{\Phi}(\boldsymbol{R},t)$
- actio $=$ reactio
  $\boldsymbol{\Phi}(\boldsymbol{n}) = -\boldsymbol{\Phi}(-\boldsymbol{n})$
- $\boldsymbol{\Psi}$, $\boldsymbol{X}$ tensor fields of the same rank n
- $\boldsymbol{\Phi}$ tensor field of the rank $n+1$

Comments (Con'd)

- formulation with respect to the mass
$$\frac{\mathrm{D}}{\mathrm{D}t}\int_m \boldsymbol{\Psi}(\boldsymbol{R},t)\,\mathrm{d}m = \frac{\mathrm{D}}{\mathrm{D}t}\int_V \boldsymbol{\Psi}(\boldsymbol{R},t)\rho(\boldsymbol{R},t)\,\mathrm{d}V$$
- from Gauss-Ostrogradsky
$$\int_A \boldsymbol{n}\cdot(\boldsymbol{\Phi})\,\mathrm{d}A = \int_V \boldsymbol{\nabla}\cdot(\boldsymbol{\Phi})\,\mathrm{d}V$$
- local form
$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho\boldsymbol{\Psi}) = \boldsymbol{\nabla}\cdot\boldsymbol{\Phi} + \rho\boldsymbol{X}$$

Balances

Balance equations are general principles for all processes.

- mass
- momentum
- angular momentum
- energy
- entropy

Balance of Mass

Conservation of Mass

$$m = \int\limits_V \rho(P,t)dV = const$$

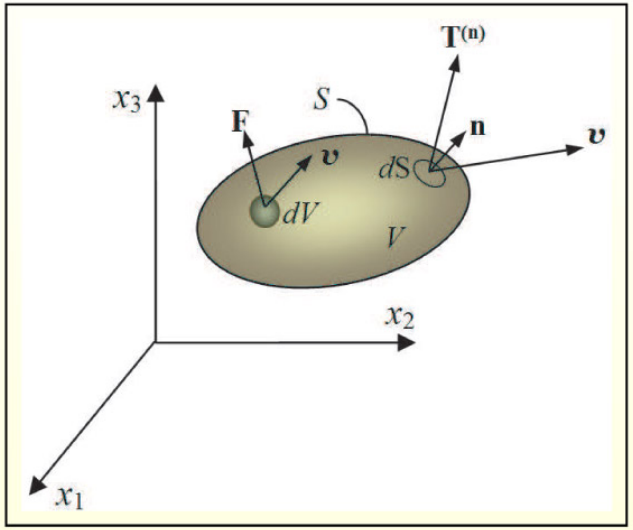
Integral form $$\frac{\mathrm{D}m}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\int\limits_V \rho(\boldsymbol{X},t)dV = \frac{\partial}{\partial t}\int\limits_{V_0} \rho_0(\boldsymbol{x})dV_0 = 0$$

Local form $$\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}m) = \frac{\mathrm{D}}{\mathrm{D}t}(\rho\ \mathrm{d}V) = \frac{\partial}{\partial t}(\rho_0\ \mathrm{d}V_0) = 0$$

Continuity equation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \quad \text{or} \quad \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\mathrm{div}\,\boldsymbol{v} = 0$$

Balance of Linear Momentum



Integral form

$$\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_V \rho(\boldsymbol{X},t)\boldsymbol{v}(\boldsymbol{X},t)dV = \int\limits_A \boldsymbol{\sigma}_{(\boldsymbol{n})}(\boldsymbol{X},t)dA + \int\limits_V \rho(\boldsymbol{X},t)\boldsymbol{F}(\boldsymbol{X},t)dV$$

Local form

$$\rho(\boldsymbol{X},t)\frac{\mathrm{D}}{\mathrm{D}t}\boldsymbol{v}(\boldsymbol{X},t) = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{X},t) + \rho(\boldsymbol{X},t)\boldsymbol{F}(\boldsymbol{X},t)$$

Balance of Angular Momentum

Integral form

$$\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_V [\boldsymbol{X} \times \rho(\boldsymbol{X},t)\boldsymbol{v}(\boldsymbol{X},t)]dV = \int\limits_A [\boldsymbol{X} \times \boldsymbol{\sigma}_{(\boldsymbol{n})}(\boldsymbol{X},\boldsymbol{n},t)]dA + \int\limits_V [\boldsymbol{X} \times \rho(\boldsymbol{X},t)\boldsymbol{F}(\boldsymbol{X},t)]dV$$

Considering the Balance of Momentum

$$\int\limits_V (\boldsymbol{I} \cdot \times \boldsymbol{\sigma})\, dV = \boldsymbol{0}$$

or the local form

$$\boldsymbol{I} \cdot \times \boldsymbol{\sigma} = \boldsymbol{0}$$

This is the symmetry of the stress tensor condition $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$

Balance of Energy - Only Mechanics

1st Law of Thermodynamics - integral form

$$\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_V \left(\frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}+u\right)\rho dV \quad = \quad \int\limits_A \boldsymbol{\sigma}_{(\boldsymbol{n})}\cdot\boldsymbol{v}dA + \int\limits_V \boldsymbol{k}\cdot\boldsymbol{v}\rho dV$$

Local form

$$\rho\dot{u} = \boldsymbol{\sigma}\cdot\cdot\left(\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{v}\right)^{\mathsf{T}} = \boldsymbol{\sigma}\cdot\cdot\boldsymbol{D}$$

First Law of Thermodynamics

General Formulation

The changes in time of the total energy W within the volume is equal to the heat flux Q and the power of all external loadings P_a.

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}W = P_a + Q}$$

$$W = U + K \qquad \text{with} \qquad \begin{array}{l} U \text{ – inner energy} \\ K \text{ – kinetic energy} \end{array}$$

First Law of Thermodynamics (con'd)

Integral Formulation

$$K = \frac{1}{2}\int\limits_V \boldsymbol{v}\cdot\boldsymbol{v}\rho\,\mathrm{d}V$$

$$U = \int\limits_m u\,\mathrm{d}m = \int\limits_V \rho u\,\mathrm{d}V$$

$$P_a = \int\limits_A \boldsymbol{t}\cdot\boldsymbol{v}\,\mathrm{d}A + \int\limits_V \boldsymbol{k}\cdot\boldsymbol{v}\rho\,\mathrm{d}V$$

$\boldsymbol{t}$ – surface traction
$\boldsymbol{k}$ – mass force

$$Q = \int\limits_V \rho r\,\mathrm{d}V - \int\limits_A \boldsymbol{n}\cdot\boldsymbol{h}\,\mathrm{d}A$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_V \left(u + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}\right)\rho\,\mathrm{d}V = \int\limits_A \boldsymbol{t}\cdot\boldsymbol{v}\,\mathrm{d}A + \int\limits_V \boldsymbol{k}\cdot\boldsymbol{v}\rho\,\mathrm{d}V =$$

$$\int\limits_A \boldsymbol{n}\cdot\boldsymbol{h}\,\mathrm{d}A + \int\limits_V \rho r\,\mathrm{d}V$$

First Law of Thermodynamics (con'd)

Some Mathematical Manipulations

$$\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_V (\ldots) = \int\limits_V \frac{\mathrm{D}}{\mathrm{D}t}(\ldots)$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}\right) = \frac{1}{2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\dot{\boldsymbol{v}}) = \dot{\boldsymbol{v}}\cdot\boldsymbol{v}$$

$$\int\limits_A \boldsymbol{n}\cdot(\boldsymbol{T}\cdot\boldsymbol{v}-\boldsymbol{h}) = \int\limits_V [\boldsymbol{\nabla}\cdot(\boldsymbol{T}\cdot\boldsymbol{v}) - \boldsymbol{\nabla}\cdot\boldsymbol{h}]\,\mathrm{d}V$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{T}\cdot\boldsymbol{v}) = (\boldsymbol{\nabla}\cdot\boldsymbol{T})\cdot\boldsymbol{v}+\boldsymbol{T}\cdot\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{v})^{\mathsf{T}} = (\boldsymbol{\nabla}\cdot\boldsymbol{T})\cdot\boldsymbol{v}+\boldsymbol{T}\cdot\cdot\boldsymbol{D}$$

First Law of Thermodynamics (con'd)

Local Form

$$\int_V \left(\frac{\mathrm{D}u}{\mathrm{D}t} + \underline{\dot{\boldsymbol{v}} \cdot \boldsymbol{v}}\right) \rho \, \mathrm{d}V =$$
$$\int_V (\boldsymbol{T} \cdot\cdot \boldsymbol{D} - \boldsymbol{\nabla} \cdot \boldsymbol{h} + \rho r) \, \mathrm{d}V + \int_V \underline{[(\boldsymbol{\nabla} \cdot \boldsymbol{T}) \cdot \boldsymbol{v} + \rho \boldsymbol{k} \cdot \boldsymbol{v}]} \, \mathrm{d}V$$

The underlined terms $\Longrightarrow$ balance of momentum

$$\int_V (\rho \dot{u} - \boldsymbol{T} \cdot\cdot \boldsymbol{D} + \boldsymbol{\nabla} \cdot \boldsymbol{h} - \rho r) \, \mathrm{d}V = 0$$

$$\boxed{\boldsymbol{T} \cdot\cdot \boldsymbol{D} - \boldsymbol{\nabla} \cdot \boldsymbol{h} + \rho r = 0}$$

Second Law of Thermodynamics

Integral and Local Formulation

$$\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_V \rho s \,\mathrm{d}V \geq \int\limits_V \frac{x}{\Theta}\rho \,\mathrm{d}V - \int\limits_A \frac{\boldsymbol{n}\cdot\boldsymbol{h}}{\Theta}\,\mathrm{d}A$$

The changes in time of the entropy within the volume under consideration is not smaller then the rate of the outer entropy flux.

$$\int\limits_A \frac{\boldsymbol{n}\cdot\boldsymbol{h}}{\Theta}\,\mathrm{d}A = \int\limits_V \boldsymbol{\nabla}\cdot\frac{\boldsymbol{h}}{\Theta}\,\mathrm{d}V = \int\limits_V \left(\frac{\boldsymbol{\nabla}\cdot\boldsymbol{h}}{\Theta} - \frac{\boldsymbol{h}\cdot\boldsymbol{\nabla}\Theta}{\Theta^2}\right)\mathrm{d}V$$

$$\frac{1}{\Theta}\boldsymbol{h}\cdot\boldsymbol{\nabla}\Theta = \boldsymbol{h}\cdot\boldsymbol{\nabla}\ln\Theta$$

$$\boxed{\rho\Theta\dot{s} \geq \rho r - \boldsymbol{\nabla}\cdot\boldsymbol{h} + \frac{1}{\Theta}\boldsymbol{h}\cdot\boldsymbol{\nabla}\Theta}$$

Dissipation Inequality

2^{nd} Law

$$\rho\Theta\dot{s} - \underline{\rho r - \boldsymbol{\nabla}\cdot\boldsymbol{h}} - \boldsymbol{h}\cdot\boldsymbol{\nabla}\ln\Theta \geq 0$$

with respect to

$$\frac{1}{\Theta}\boldsymbol{h}\cdot\boldsymbol{\nabla}\Theta = \boldsymbol{h}\cdot\boldsymbol{\nabla}\ln\Theta$$

1^{st} Law

$$\rho\dot{s} = \boldsymbol{T}\cdot\cdot\boldsymbol{D} - \underline{\boldsymbol{\nabla}\cdot\boldsymbol{h} + \rho r}$$

$$\Longrightarrow\quad \rho\Theta\dot{s} + \boldsymbol{T}\cdot\cdot\boldsymbol{D} - \rho\dot{s} + \boldsymbol{h}\cdot\boldsymbol{\nabla}\ln\Theta \geq 0$$

Dissipation Inequality (con'd)

$$\rho\Theta\dot{s} = \rho\left(\Theta s\right)^{\cdot} - \rho s\dot{\Theta}$$

$$\Longrightarrow \quad \rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\Theta s - u\right) - \rho s\frac{\mathrm{D}\Theta}{\mathrm{D}t} + \boldsymbol{T}\cdot\cdot\boldsymbol{D} - \boldsymbol{h}\cdot\nabla\ln\Theta \geq 0$$

Helmholtz' Free Energy

$$u - \Theta s = f$$

$$\Longrightarrow \quad \boldsymbol{T}\cdot\cdot\boldsymbol{D} - \rho\frac{\mathrm{D}f}{\mathrm{D}t} - \rho s\frac{\mathrm{D}\Theta}{\mathrm{D}t} - \boldsymbol{h}\cdot\nabla\ln\Theta \geq 0$$

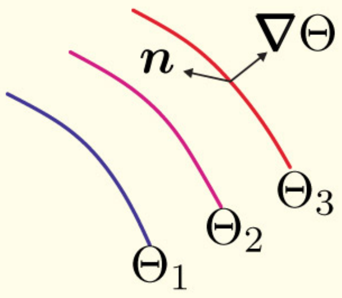
Dissipation Function

$$\boldsymbol{T}\cdot\cdot\boldsymbol{D} - \rho\left(\dot{f} + s\dot{\Theta}\right) = \Phi \geq 0)$$

Heat Flux and Temperature Field

$$\boldsymbol{h} \cdot \boldsymbol{\nabla} \ln \Theta \geq 0 \quad \text{or} \quad \frac{\boldsymbol{h}}{\Theta} \cdot \boldsymbol{\nabla} \Theta \geq 0 \quad \text{with} \quad \Theta > 0$$

- $\boldsymbol{h} = \boldsymbol{0}$ adiabatic process
- $\boldsymbol{\nabla} \Theta = \boldsymbol{0}$ isothermal process



$$\Theta_1 < \Theta_2 < \Theta_3$$

- non-dissipative process

$$\varPhi = 0$$

$$\angle (\boldsymbol{h}, \boldsymbol{\nabla\Theta}) > \frac{\pi}{2}$$

exception: orthogonality

Heat Transfer

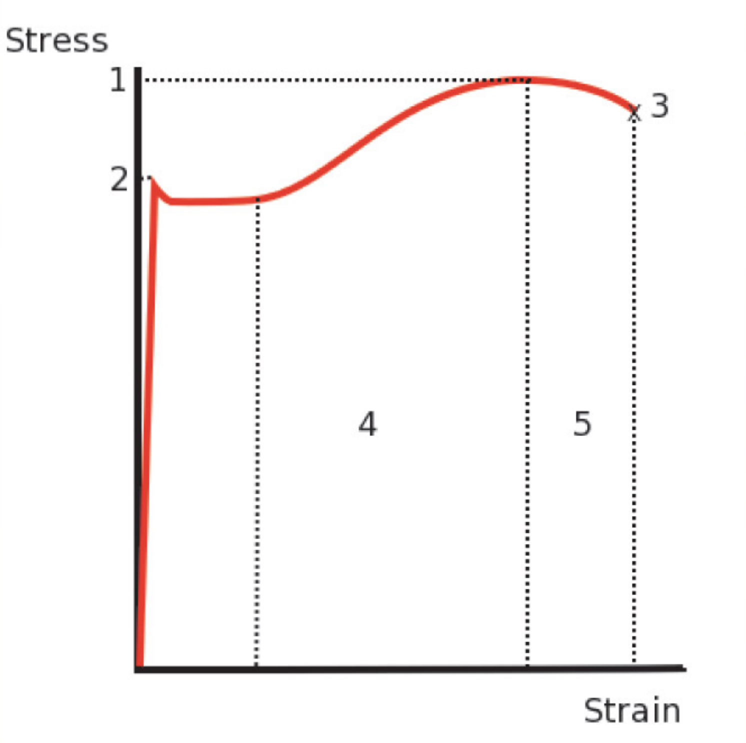
1st Law

$$\rho\Theta\frac{\mathrm{D}s}{\mathrm{D}t} = \boldsymbol{T}\cdot\cdot\boldsymbol{D} - \rho\left(\frac{\mathrm{D}f}{\mathrm{D}t} + s\frac{\mathrm{D}\Theta}{\mathrm{D}t}\right) + \rho r - \boldsymbol{\nabla}\cdot\boldsymbol{h} = \Phi + \rho r - \boldsymbol{\nabla}\cdot\boldsymbol{h}$$

- *non-dissipative process:* $\Phi = 0$
  $\rho\Theta\frac{\mathrm{D}s}{\mathrm{D}t} = \rho r - \boldsymbol{\nabla}\cdot\boldsymbol{h}$ heat transfer
- *isothermal process:* no heat transfer, mechanical and thermal processes are decoupled
- *adiabatic process:* $\boldsymbol{h} = \boldsymbol{0}$, $r = 0$

Constitutive Equations

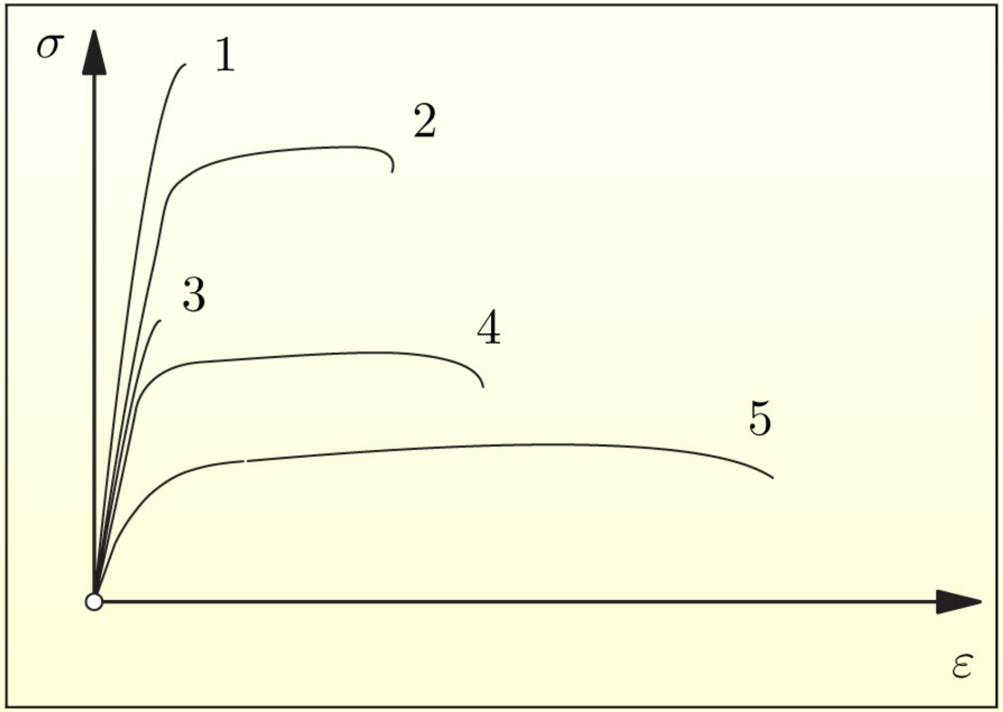
Stress-strain Curve for Low-carbon Steel



1. Ultimate strength, 2. Yield strength-corresponds to yield point, 3. Rupture, 4. Strain hardening region, 5. Necking region

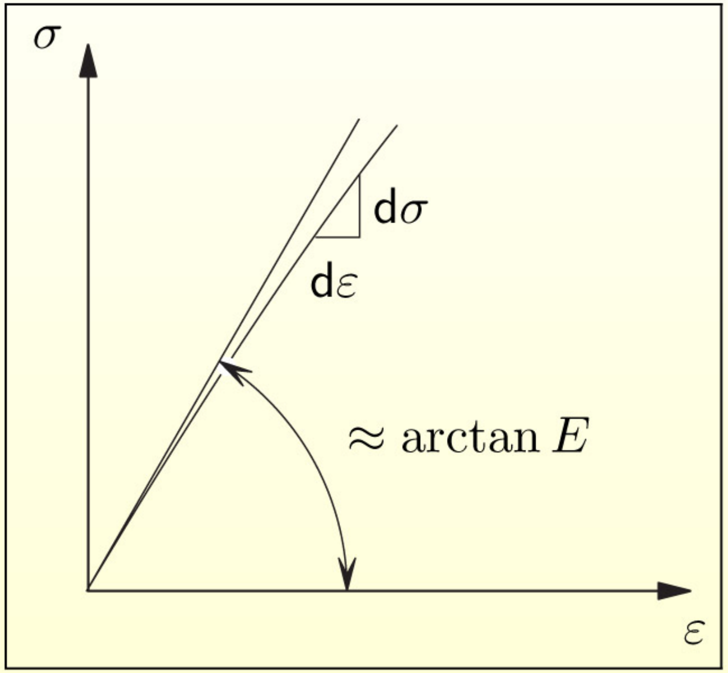
Hooke's law is only valid for the portion of the curve between the origin and the yield point.

Stress-strain Diagrams of Various Metals



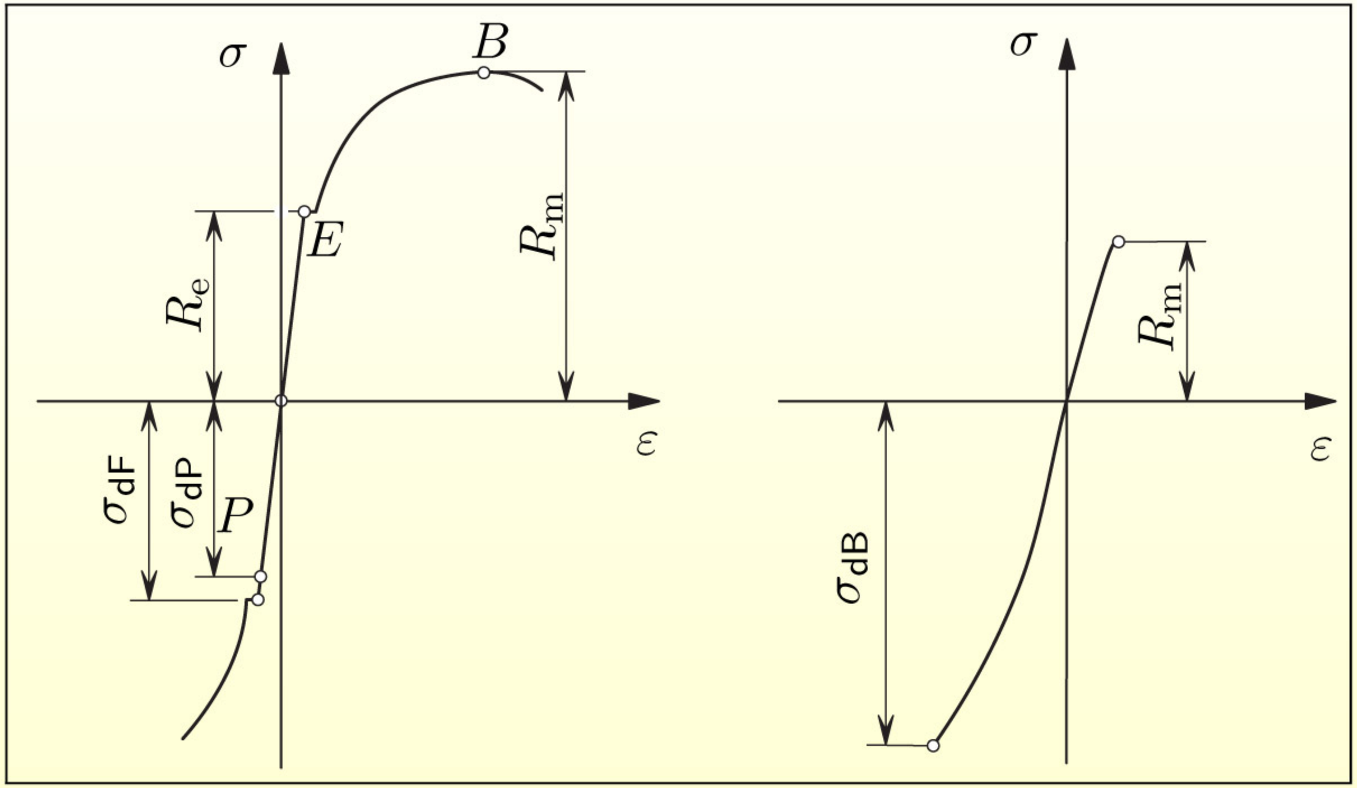
1 hardened steel, 2 tempered (high yield) steel, 3 cast iron, 4 aluminium alloy, 5 pure copper

Nonlinear Elastic Behavior



Example: cast iron, plastics

Different Behavior at Tension and Compression



steel cast iron

Constitutive Equations

Approaches

Materials Physics

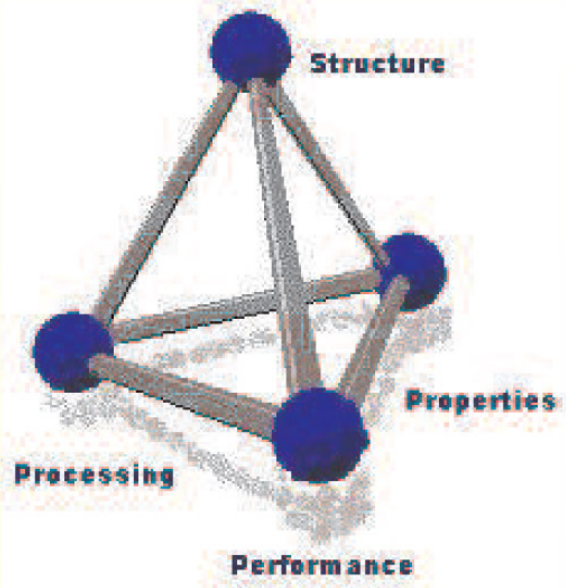
Material physics

is the use of physics to describe materials. It is a synthesis of physical sciences such as chemistry, solid mechanics and solid state physics.

Solid-state physics

the largest branch of condensed matter physics, is the study of rigid matter, or solids. The bulk of solid-state physics theory and research is focused on crystals, largely because the periodicity of atoms in a crystal – its defining characteristic – facilitates mathematical modeling, and also because crystalline materials often have electrical, magnetic, optical, or mechanical properties that can be exploited for engineering purposes.

Materials Science



Materials science or **materials engineering** is an interdisciplinary field involving the properties of matter and its applications to various areas of science and engineering. This science investigates the **relationship between the structure of materials and their properties**. It includes elements of applied physics and chemistry, as well as chemical, mechanical, civil and electrical engineering.

Material Dependent Equations

- **Specific (individual) response** of the given material on arbitrary load.
- Modeling principles
  - **Inductive approach**
    from the simplest to more complex models
  - **Deductive approach**
    from the general frame to special cases
- Identification
  - Experimental observations
  - Mathematical analysis
  - Theory of symmetry (Curie-Neumann's principle)

Theory of Materials[26]

Basic items

- Formulation of suitable constitutive and evolution equations
- Checking the correctness of the formulation and the adequateness to thermodynamic considerations
- Experimental identification of the parameters, etc.

Conclusions

- Formulation of constitutive equations
- Including material symmetries
- Including constraints

[26] Haupt, P. Continuum Mechanics and Theory of Materials, 2nd edition. Springer, Berlin, 2002

Basic Definitions (I)

Definition

Constitutive equations connecting all macroscopic phenomenological variables describing the behavior of the continuum.

Definition

Simple materials of the rank 1 are materials which are described bei constitutive equations connecting local variables, e.g. the local strain tensor and the local heat flux vector with the local stress tensor and the local temperature gradient. All statements are related to the same material point and its differential neighborhood of rank 1.

Definition

Process is the change of the constitutive parameters with respect of time.

Basic Definitions (II)

Definition

Behavior of the continuum in each material point is given as a set of constitutive variables which are operators with respect to time.

Definition

Solid is a material behavior if at given loads the stress deviator has non-zero components, that means it shows resistance if the shape is changing.

Definition

Fluid is a material behavior if at given loads the stress deviator has only zero components, that means it not preserves the shape.

Axioms of the Materials Theory

- Causality
- Determinism
- Equipresence
- Material objectivity
- Local action
- Memory
- Physical consistency

Simple Thermomechanical Material

General Form

$$
\begin{aligned}
\boldsymbol{P}(\boldsymbol{x},t) &= \boldsymbol{P} \quad \{\boldsymbol{x},\theta(\boldsymbol{x},t),\dot{\theta}(\boldsymbol{x},t),\boldsymbol{\nabla}_{\boldsymbol{x}}\theta(\boldsymbol{x},t),\boldsymbol{\nabla}_{\boldsymbol{x}}\dot{\theta}(\boldsymbol{x},t),\boldsymbol{\Gamma}(\boldsymbol{x},t)\}\\
\boldsymbol{h}_0(\boldsymbol{x},t) &= \boldsymbol{h}_0 \quad \{\boldsymbol{x},\theta(\boldsymbol{x},t),\dot{\theta}(\boldsymbol{x},t),\boldsymbol{\nabla}_{\boldsymbol{x}}\theta(\boldsymbol{x},t),\boldsymbol{\nabla}_{\boldsymbol{x}}\dot{\theta}(\boldsymbol{x},t),\boldsymbol{\Gamma}(\boldsymbol{x},t)\}\\
f(\boldsymbol{x},t) &= f \quad \{\boldsymbol{x},\theta(\boldsymbol{x},t),\dot{\theta}(\boldsymbol{x},t),\boldsymbol{\nabla}_{\boldsymbol{x}}\theta(\boldsymbol{x},t),\boldsymbol{\nabla}_{\boldsymbol{x}}\dot{\theta}(\boldsymbol{x},t),\boldsymbol{\Gamma}(\boldsymbol{x},t)\}\\
s(\boldsymbol{x},t) &= s \quad \{\boldsymbol{x},\theta(\boldsymbol{x},t),\dot{\theta}(\boldsymbol{x},t),\boldsymbol{\nabla}_{\boldsymbol{x}}\theta(\boldsymbol{x},t),\boldsymbol{\nabla}_{\boldsymbol{x}}\dot{\theta}(\boldsymbol{x},t),\boldsymbol{\Gamma}(\boldsymbol{x},t)\}
\end{aligned}
$$

Replacement of $\boldsymbol{\Gamma}$

$$
\begin{aligned}
\boldsymbol{P}(\boldsymbol{x},t) &= \boldsymbol{P} \quad \{\boldsymbol{x},\theta,\dot{\theta},\boldsymbol{\nabla}_{\boldsymbol{x}}\theta,\boldsymbol{\nabla}_{\boldsymbol{x}}\dot{\theta},\boldsymbol{C},\dot{\boldsymbol{C}},\rho^{-1},\dot{\rho}\}\\
\boldsymbol{h}_0(\boldsymbol{x},t) &= \boldsymbol{h} \quad \{\boldsymbol{x},\theta,\dot{\theta},\boldsymbol{\nabla}_{\boldsymbol{x}}\theta,\boldsymbol{\nabla}_{\boldsymbol{x}}\dot{\theta},\boldsymbol{f},\dot{\boldsymbol{C}},\rho^{-1},\dot{\rho}\}\\
f(\boldsymbol{x},t) &= f \quad \{\boldsymbol{x},\theta,\dot{\theta},\boldsymbol{\nabla}_{\boldsymbol{x}}\theta,\boldsymbol{\nabla}_{\boldsymbol{x}}\dot{\theta},\boldsymbol{C},\dot{\boldsymbol{C}},\rho^{-1},\dot{\rho}\}\\
s(\boldsymbol{x},t) &= s \quad \{\boldsymbol{x},\theta,\dot{\theta},\boldsymbol{\nabla}_{\boldsymbol{x}}\theta,\boldsymbol{\nabla}_{\boldsymbol{x}}\dot{\theta},\boldsymbol{C},\dot{\boldsymbol{C}},\rho^{-1},\dot{\rho}\}
\end{aligned}
$$